Introduction to Dynamical Systems

Solutions Problem Set 10

Exercise 1. Let $A \in \mathrm{Mat}(n \times n, \mathbb{R})$ a matrix whose eigenvalues $\lambda \in \mathbb{C}$ satisfy

$$\alpha < \operatorname{Re}\lambda < \beta$$

for two real numbers α, β. Show that there is an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\alpha \left\|x\right\|^2 \leq \langle Ax, x \rangle \leq \beta \left\|x\right\|^2 ,$$

where $\left\|x\right\|^2 = \langle x, x \rangle$.

Solution. Let c be such that $\operatorname{Re}\lambda < c < \beta$ for all eigenvalue λ of A. We may work in the basis which represents A in its real canonical form, as composition with a change-of-basis matrix makes an inner product remain an inner product. Furthermore, we may assume that A consists of a single block, as proving the result for a block yields the result for a general matrix consisting of various blocks. These can be either of the form

$$A = \begin{bmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & & \\ & & \lambda & & & \\ & & & \ddots & 1 & \\ & & & & \lambda & 1 \\ & & & & & \lambda \end{bmatrix} = \lambda I + N, \tag{1}$$

or

$$A = \begin{bmatrix} D & I_2 & & & & \\ & D & I_2 & & & \\ & & D & & & \\ & & & \ddots & I_2 & \\ & & & & D & I_2 \\ & & & & & D \end{bmatrix}, \quad D = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2}$$

If A is of the form (1), then the basis vectors $\{e_1, \dots, e_n\}$ satisfy $Ne_j = e_{j+1}$ for $1 \leq j \leq n-1$, and $Ne_n = 0$, and letting $\varepsilon > 0$ we may consider a different basis

$$\mathcal{B}_\varepsilon = \left\{ e_1, \frac{1}{\varepsilon}e_2, \dots, \frac{1}{\varepsilon^{n-1}}e_n \right\} = \{u_1, \dots u_n\}$$

which now satisfies $Nu_j = \varepsilon u_{j+1}$ for $1 \leq j \leq n-1$, and $Nu_n = 0$. Therefore, the matrix A in the new basis $\mathcal{B}_\varepsilon$ corresponds to

$$A_\varepsilon = \begin{bmatrix} \lambda & \varepsilon & & & & \\ & \lambda & \varepsilon & & & \\ & & \lambda & & & \\ & & & \ddots & \varepsilon & \\ & & & & \lambda & \varepsilon \\ & & & & & \lambda \end{bmatrix}.$$

Let now $\langle \cdot, \cdot \rangle_\varepsilon$ be the inner product corresponding to the basis $\mathcal{B}_\varepsilon$. Then

$$\frac{\langle Ax, x \rangle_\varepsilon}{\langle x, x \rangle_\varepsilon} \longrightarrow \frac{\langle \lambda I x, x \rangle}{\langle x, x \rangle}, \quad \text{as } \varepsilon \to 0.$$

Letting ε sufficiently small, the basis satisfies the lemma for a block of the form (1).

For blocks of the form (2) the idea remains the same, and by modifying the basis slightly one shows the same result. The case where A is semisimple is easy to deal with, as it follows from direct computations since there are no N or I_2 blocks. Finally, the lower bound follows exactly by the same argument but considering $\alpha < c < \operatorname{Re} \lambda$. $\square$

Exercise 2. Consider the system of ODEs

$$\dot{y} = f(y)$$

where as usual $y = (y_1, \ldots, y_n)$. Assume that $0 \in \mathbb{R}^n$ is a fixed point, i.e. $f(0) = 0$, and $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Finally, assume that $Df(0)$, the Jacobian matrix, only has eigenvalues with negative real part. By using **Exercise 1**, show directly (i.e. without using the Hartman-Grobman theorem) that there exist $C > 0$, $c > 0$ and $\delta > 0$ such that for any initial condition

$$y(0), |y(0)| < \delta,$$

we have that

$$|y(t)| \leq C e^{-ct} |y(0)|.$$

Solution. Notice that y is given by the Banach fixed point theorem for $t \in [0, 1]$ thanks to

$$y(t) = \int_0^t e^{(t-s)Df(0)} F(y(s)) \, \mathrm{d}s + e^{tDf(0)} y(0), \quad |F(y)| = o(|y|)$$

(cf. Lecture9.pdf Lemma 3.3). By noting that now $Df(0)$ has strictly negative eigenvalues, we have that **Exercise 1** holds in the Euclidean norm up to a constant, and we can estimate using the triangle inequality to show that, whenever $t < T \subset \mathbb{R}$ is small enough such that $F(y(s))$ can be bounded by $r |y(0)|$ for a small $r > 0$, we have

$$|y(t)| \leq K e^{-\kappa t} |y(0)| + K r e^{-\kappa t} \int_0^t e^{\kappa s} |y(s)| \, \mathrm{d}s$$

By rearranging and using that $|y(0)| < \delta$, we reach

$$e^{\kappa t} |y(t)| \leq K\delta + Kr \int_0^t e^{\kappa s} |y(s)| \, \mathrm{d}s.$$

At this point we may use Gronwall's inequality to show that

$$e^{\kappa t} |y(t)| \leq K \delta e^{Krt}.$$

Rearranging again,

$$|y(t)| \leq K \delta e^{(Kr - \kappa)t}.$$

Now, choosing r small enough such that $Kr - \kappa = -c < 0$ and letting δ be small enough so that $|F(y)| \leq r |y|$ for all $|y| < K\delta$, we conclude that

$$|y(t)| \leq C e^{-ct} |y(0)|$$

for small times $t < T$. The bound is then easily extended for large times by noting that $|y(t)|$ decays and therefore $|F(y(s))| \leq r |y(s)|$ for all $s \geq 0$. $\square$

Exercise 3. Give an example of a *non-diophantine number* $\alpha \in \mathbb{R}$ in the sense of Lecture10.pdf.

Solution. We look for $\alpha \in \mathbb{R}$ for which for all $c > 0$ and $d > 0$ there is $(p, q) \in \mathbb{Z}_+ \times \mathbb{Z}$ with

$$\left|\alpha - \frac{p}{q}\right| \leq c\,|p|^{-(d+1)}.$$

An example of such a number is an appropriate modification of Liouville's constant. More generally, Liouville numbers can be constructed by letting $b \geq 2$ and choosing $(a_1, a_2, \dots)$ such that $0 \leq a_k \leq b - 1$ and $a_k \neq 0$ for infinitely many k, and then setting

$$x = \sum_{k \geq 1} \frac{a_k}{b^{k!}}.$$

Since x has a non-repeating base b representation, it follows that it is not a rational number, and for any $d, c > 0$, we choose $n = \lceil d \rceil$ and define p_n and q_n as

$$q_n = b^{\frac{n}{n+1} n!} c^{1/n}, \quad p_n = q_n \sum_{k=1}^{n} \frac{a_k}{b^{k!}}.$$

Then, we estimate

$$\begin{aligned} 0 < \left|x - \frac{p_n}{q_n}\right| &= \left|x - \sum_{k=1}^{n} \frac{a_k}{b^{k!}}\right| = \left|\sum_{k=1}^{\infty} \frac{a_k}{b^{k!}} - \sum_{k=1}^{n} \frac{a_k}{b^{k!}}\right| = \left|\sum_{k=n+1}^{\infty} \frac{a_k}{b^{k!}}\right| \leq \sum_{k=n+1}^{\infty} \frac{b-1}{b^{k!}} \\ &= \frac{b-1}{b^{(n+1)!}} \cdot \sum_{k=0}^{\infty} \frac{1}{b^k} = \frac{b-1}{b^{(n+1)!}} \cdot \frac{b}{b-1} = \frac{b}{b^{(n+1)!}} \leq \frac{b^{n!}}{b^{(n+1)!}} = \frac{1}{b^{(n+1)!-n!}} = \frac{c}{q_n^{n+1}} \leq \frac{c}{q_n^{d+1}}. \end{aligned}$$

$\square$